A Prudent Logic of Partial Functions as a Unifying Framework of Many Sorted Logics

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- Many Sorted Logic and Partial Functions Logic
- Reduction results
- Order Sorted Logic and Inductive Data Types
- Conclusion

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Many Sorted Logic and Partial Functions Logic

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Many Sorted Logic – FO(TY):

- Reasoning about different sorts of objects.
- E.g., Mortality applies only to living beings.
- Hierarchy of sorts (E.g., Animals :> (Cats, Dogs, ...)).
- Syntactical decidability of well-typed formulae.

Partial Functions Logic – FO(PF):

- E.g., Subtraction in Natural numbers (i.e., $5 - 10$).
- E.g., Division in Natural and Real numbers (i.e., $5/0$).
- E.g., The present king of France is bald.
- E.g., Next element of a list.

On denoting. Bertrand Russell. Mind, 1905

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FO(TY) & FO(PF) – Syntax

FO(TY):

$$
\begin{aligned}
\tau &:= x \\
&:= f(\tau_1, \ldots, \tau_n)
\end{aligned}
$$

$$
\begin{aligned}
\phi &:= p(\tau_1, \ldots, \tau_n) \\
&:= \neg \phi_1 \\
&:= \phi_1 \lor \phi_2 \\
&:= \exists x[\mathbb{T}] : \phi_1
\end{aligned}
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FO(PF):

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FO(TY) Well typed formulae

- Typical for sorted logic is type checking inference (is formula well-typed or not).
- What it means for a formula to be well typed?
- Types of terms are matching the type of argument where they are applied.
- This eliminates formulae that are trivially tautologies or contradictions.
- Helps KB engineers in reducing logical errors.

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FO(TY) Well typed formulae

Vocabulary:

$$\mathit{type}\ \mathit{Cat}$$
$$\mathit{type}\ \mathit{Dog}$$
$$\mathit{type}\ \mathit{Human}$$
$$\mathit{type}\ \mathit{Rock}$$

$$\mathit{Garfield} : () \to \mathit{Cat}$$
$$\mathit{Meows} : \mathit{Cat} \to \mathbb{B}$$
$$\mathit{Barks} : \mathit{Dog} \to \mathbb{B}$$
$$\mathit{Mortal} : \mathit{Human} \to \mathbb{B}$$

Is formula well typed:

$$\mathit{Meows}(\mathit{Garfield})$$
$$\mathit{Barks}(\mathit{Garfield})$$
$$\forall r[\mathit{Rock}] : \mathit{Mortal}(r)$$
$$\forall h[\mathit{Human}] : \mathit{Mortal}(h)$$

FO(TY) Well typed formulae

$$\frac{}{\omega \vdash \mathsf{t} : \mathbb{B}} \; (\textit{T-tr}) \qquad \frac{}{\omega \vdash \mathsf{f} : \mathbb{B}} \; (\textit{T-fa}) \qquad \frac{\omega \vdash \phi : \mathbb{B}}{\omega \vdash \neg\phi : \mathbb{B}} \; (\textit{T-neg})$$

$$\frac{\omega \vdash \phi : \mathbb{B} \qquad \omega \vdash \varphi : \mathbb{B}}{\omega \vdash (\phi \vee \varphi) : \mathbb{B}} \; (\textit{T-or}) \qquad \frac{\omega \cup \{x : \mathbb{T}\} \vdash \phi : \mathbb{B}}{\omega \vdash (\exists x[\mathbb{T}] : \phi) : \mathbb{B}} \; (\textit{T-ex})$$

$$\frac{x : \mathbb{T} \in \omega}{\omega \vdash x : \mathbb{T}} \; (\textit{T-v}) \qquad \frac{s_\Sigma(\sigma) = (\mathbb{T}_1, \ldots, \mathbb{T}_n, \mathbb{T}) \qquad \omega \vdash t_i : \mathbb{T}_i}{\omega \vdash \sigma(t_1, \ldots, t_n) : \mathbb{T}} \; (\textit{T-a})$$

FO(TY) Well typed formulae

Vocabulary:

$type\ Cat$

$type\ Dog$

$type\ Human$

$type\ Rock$

$Garfield : () \rightarrow Cat$

$Meows : Cat \rightarrow \mathbb{B}$

$Barks : Dog \rightarrow \mathbb{B}$

$Mortal : Human \rightarrow \mathbb{B}$

Well typed formula:

$Meows(Garfield)$

$\forall h[Human] : Mortal(h)$

Ill typed formula:

$Barks(Garfield)$

$\forall r[Rock] : Mortal(r)$

FO(PF) Well guarded formulae

- Typical for partial functions logic is well definedness checking inference (does formula evaluate to true or false in all structures).
- However, this is in general undecidable!
- We introduce notion of well guarded formula.
- Informally, formula is well guarded if all partial function terms are constrained with their domain.
- Well guarded formulae are well defined!

A practical approach to partial functions in CVC lite. Berezin, S., Barrett, C., Shikanian, I., Chechik, M., Gurfinkel, A., Dill, D.L. Electronic Notes in Theoretical Computer Science, 2005

Towards Systematic Treatment of Partial Functions in Knowledge Representation. Djordje Markovic, Maurice Bruynooghe, Marc Denecker. Logics in Artificial Intelligence JELIA 2023

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FO(PF) Well guarded formulae

Vocabulary:

$$Doctor : \mathbb{U} \to \mathbb{B}$$
$$Mother : \mathbb{U} \nrightarrow \mathbb{U}$$
$$MyPartner : () \nrightarrow \mathbb{U}$$

$$\delta_{Mother} : \mathbb{U} \to \mathbb{B}$$
$$\delta_{MyPartner} : () \to \mathbb{B}$$

Is formula well guarded:

$$Doctor(Mother(MyPartner))$$

$$\textbf{if } \delta_{MyPartner}() \textbf{ then } Doctor(Mother(MyPartner)) \textbf{ else } \mathrm{f} \textbf{ fi}$$

$$
\begin{array}{l}
\textbf{if } \delta_{MyPartner}() \textbf{ then} \\
\quad \textbf{if } \delta_{Mother}(MyPartner) \textbf{ then} \\
\quad\quad Doctor(Mother(MyPartner)) \\
\quad \textbf{else } \mathrm{f} \textbf{ fi} \\
\textbf{else } \mathrm{f} \textbf{ fi}
\end{array}
$$

FO(PF) Well guarded formulae

$$\frac{}{\gamma \Vdash \mathsf{t}}\ \textit{(G-tr)} \quad \frac{}{\gamma \Vdash \mathsf{f}}\ \textit{(G-fa)} \quad \frac{}{\gamma \Vdash x}\ \textit{(G-v)} \quad \frac{\gamma \Vdash \phi}{\gamma \Vdash \neg\phi}\ \textit{(G-neg)}$$

$$\frac{\gamma \Vdash \phi \qquad \gamma \Vdash \varphi}{\gamma \Vdash (\phi \lor \varphi)}\ \textit{(G-or)} \quad \frac{\gamma \Vdash \phi}{\gamma \Vdash (\exists x : \phi)}\ \textit{(G-ex)} \quad \frac{\delta_\sigma(\bar{t}) \in \gamma \qquad \gamma \Vdash \bar{t}}{\gamma \Vdash \sigma(\bar{t})}\ \textit{(G-a)}$$

$$\frac{\gamma \Vdash p_1(\bar{t}_1) \ldots \gamma \Vdash p_n(\bar{t}_n) \qquad \gamma \Vdash \phi \qquad \omega \cup \{p_1(\bar{t}_1), \ldots, p_n(\bar{t}_n)\} \Vdash \psi \qquad \gamma \Vdash \chi}{\gamma \Vdash \textbf{if } p_1(\bar{t}_1) \land \cdots \land p_n(\bar{t}_n) \land \phi \textbf{ then } \psi \textbf{ else } \chi \textbf{ fi}}\ \textit{(G-g)}$$

FO(PF) Well guarded formulae

Vocabulary:

$$\begin{aligned} &\mathit{Doctor} : \mathbb{U} \to \mathbb{B} \\ &\mathit{Mother} : \mathbb{U} \nrightarrow \mathbb{U} \\ &\mathit{MyPartner} : () \nrightarrow \mathbb{U} \end{aligned}$$

$$\begin{aligned} &\delta_{\mathit{Mother}} : \mathbb{U} \to \mathbb{B} \\ &\delta_{\mathit{MyPartner}} : () \to \mathbb{B} \end{aligned}$$

Unguarded formulae:

$$\mathit{Doctor}(\mathit{Mother}(\mathit{MyPartner}))$$

$$\textbf{if } \delta_{\mathit{MyPartner}}() \textbf{ then } \mathit{Doctor}(\mathit{Mother}(\mathit{MyPartner})) \textbf{ else } \mathsf{f} \textbf{ fi}$$

Well guarded formula:

$$\begin{aligned} &\textbf{if } \delta_{\mathit{MyPartner}}() \textbf{ then} \\ &\quad \textbf{if } \delta_{\mathit{Mother}}(\mathit{MyPartner}) \textbf{ then} \\ &\qquad \mathit{Doctor}(\mathit{Mother}(\mathit{MyPartner})) \\ &\quad \textbf{else } \mathsf{f} \textbf{ fi} \\ &\textbf{else } \mathsf{f} \textbf{ fi} \end{aligned}$$

FO(PF) Well guarded formulae

Vocabulary:

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$$MyPartner : () \nrightarrow \mathbb{U}$$

$$\delta_{Mother} : \mathbb{U} \to \mathbb{B}$$
$$\delta_{MyPartner} : () \to \mathbb{B}$$

Is this theory well guarded:

$$\delta_{MyPartner}()$$
$$\delta_{Mother}(MyPartner)$$
$$Doctor(Mother(MyPartner))$$

FO(PF) Well guarded formulae

$$\frac{\delta_\sigma(\bar{t}) \mathrel{\underline{\in}} \gamma, \overline{\mathcal{T}} \qquad \gamma, \overline{\mathcal{T}} \Vdash \bar{t}}{\gamma, \overline{\mathcal{T}} \Vdash \sigma(\bar{t})} \textit{(G-a')} \qquad \frac{\forall \bar{x} : p(\bar{r}) \land \phi \in \overline{\mathcal{T}}}{p(\bar{r}[\bar{x} \to \bar{u}]) \mathrel{\underline{\in}} \gamma, \overline{\overline{\mathcal{T}}}} \textit{(G-c)} \qquad \frac{p(\bar{r}) \in \gamma}{p(\bar{r}) \mathrel{\underline{\in}} \gamma, \overline{\overline{\mathcal{T}}}} \textit{(G-c')}$$

$$\frac{\forall \bar{x} : p_1(\bar{s}_1) \land \cdots \land p_m(\bar{s}_m) \Rightarrow p(\bar{r}) \land \phi \in \overline{\mathcal{T}} \qquad p_i(\bar{s}_i)[\bar{x} \to \bar{u}] \mathrel{\underline{\in}} \gamma, \overline{\mathcal{T}}}{p(\bar{r}[\bar{x} \to \bar{u}]) \mathrel{\underline{\in}} \gamma, \overline{\mathcal{T}}} \textit{(G-i)}$$

FO(PF) Well guarded formulae

Vocabulary:

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\begin{aligned}
&\mathit{Doctor} : \mathbb{U} \rightarrow \mathbb{B}\\
&\mathit{Mother} : \mathbb{U} \nrightarrow \mathbb{U}\\
&\mathit{MyPartner} : () \nrightarrow \mathbb{U}
\end{aligned}
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$$
\begin{aligned}
&\delta_{\mathit{Mother}} : \mathbb{U} \rightarrow \mathbb{B}\\
&\delta_{\mathit{MyPartner}} : () \rightarrow \mathbb{B}
\end{aligned}
$$

This theory is well guarded:

$$
\begin{aligned}
&\delta_{\mathit{MyPartner}}()\\
&\delta_{\mathit{Mother}}(\mathit{MyPartner})\\
&\mathit{Doctor}(\mathit{Mother}(\mathit{MyPartner}))
\end{aligned}
$$

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FO(TY) to FO(PF) Reduction results

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- Can we define translation of FO(TY) to FO(PF) such that:
- FO(TY) theory is well typed iff its transition is well guarded?
- We can recover models of FO(TY) theory from the models of its translation to FO(PF)?
- The answer is yes!

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FO(TY) to FO(PF) Reduction Results

FO(TY):	FO(PF):	
$\mathit{type}\ \mathit{Cat}$	$\mathit{Cat} : \mathbb{U} \nrightarrow \mathbb{B}$	$\{\forall x : \delta_{\mathit{Cat}}(x) \qquad \exists x : \mathit{Cat}(x)\}$
$\mathit{Garfield} : () \rightarrow \mathit{Cat}$	$\mathit{Garfield} : () \nrightarrow \mathbb{U}$	$\{\delta_{\mathit{Garfield}}() \qquad \delta_{\mathit{Garfield}}() \Rightarrow \mathit{Cat}(\mathit{Garfield})\}$
$\mathit{Meows} : \mathit{Cat} \rightarrow \mathbb{B}$	$\mathit{Meows} : \mathbb{U} \nrightarrow \mathbb{B}$	$\{\forall x : \delta_{\mathit{Meows}}(x) \Leftrightarrow \mathit{Cat}(x)\}$
$\mathit{Meows}(\mathit{Garfield})$		$\{\mathit{Meows}(\mathit{Garfield})\}$
$\exists c[\mathit{Cat}] : \mathit{Meows}(c)$		$\{\exists c : \textbf{if}\ \mathit{Cat}(c)\ \textbf{then}\ \mathit{Meows}(c)\ \textbf{else}\ \mathrm{f}\ \textbf{fi}\}$

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Order Sorted Logic and Inductive Data Types

Order Sorted Logic and Inductive Data Types

- Can we achieve the same results for Order Sorted Logic as for FO(TY)?
- What is Sorted Logic that corresponds to the full generality of FO(PF)?
- FO(PF) also supports inductive definitions, can we do more with them?

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Order Sorted Logic

Vocabulary:

$$\begin{array}{l} type\ Animal \\ type\ Cat <: Animal \\ type\ Dog <: Animal \end{array}$$

$$\begin{array}{l} Meows : Cat \to \mathbb{B} \\ Barks : Dog \to \mathbb{B} \\ Mortal : Animal \to \mathbb{B} \end{array}$$

Is formula well typed:

$$\begin{array}{l} \exists c[Cat] : Meows(c) \\ \exists d[Dog] : Barks(c) \\ \forall d[Dog] : Mortal(d) \end{array}$$

Beyond Order Sorted Logic

Vocabulary:

$$\begin{aligned}&\mathit{type}\ \mathit{Animal}\\&\mathit{type}\ \mathit{Cat} <: \mathit{Animal}\\&\mathit{type}\ \mathit{Dog} <: \mathit{Animal}\\[1em]&\mathit{Meows} : \mathit{Cat} \to \mathbb{B}\\&\mathit{Barks} : \mathit{Dog} \to \mathbb{B}\end{aligned}$$

Is formula well typed:

$$\begin{aligned}&\exists a[\mathit{Animal}] :\\&\quad \textbf{if } \mathit{Cat}(a) \textbf{ then } \mathit{Meows}(a) \textbf{ else}\\&\qquad\quad \textbf{if } \mathit{Dog}(a) \textbf{ then}\\&\qquad\qquad\quad \mathit{Barks}(a)\\&\qquad\quad \textbf{else } \mathrm{f} \textbf{ fi}\\&\quad\ \textbf{fi}\end{aligned}$$

Inductive Data Types

Vocabulary:

$$\textit{type Nat where}$$
$$z : Nat$$
$$s : Nat \to Nat$$

In FO(PF):

$$z : () \nrightarrow \mathbb{U}$$
$$s : \mathbb{U} \nrightarrow \mathbb{U}$$
$$Nat : \mathbb{U} \to \mathbb{B}$$

$$\left\{ \begin{array}{l} Nat(z). \\ \forall x : Nat(s(x)) \leftarrow Nat(x). \end{array} \right\}$$

$$\delta_z()$$
$$\forall x : \delta_s(x) \Leftrightarrow Nat(x)$$
$$UNA(z, s)$$

Content

- ~~Many Sorted Logic and Partial Functions Logic~~
- ~~Reduction results~~
- ~~Order Sorted Logic and Inductive Data Types~~
- Conclusion

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Thank you for your attention

Typed relations/functions can be thought of as partial relations/functions on the universe, where well guarded relation generalizes the idea of well typed relation.

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